Introduction to astrophysical plasma physics: relevant scales and equations

based on R. Kulsrud „Plasma Physics for Astrophysics"

- What is plasma? What is the most general way of describing the plasma?
  - Distribution function
  - Kinetic equation, Maxwell's equations
  - Collisions
- Particle motion in uniform fields
  - Drifts
  - Larmor radius
  - Adiabatic invariants
- Basic collective plasma processes
  - Plasma frequency, skin depth (inertial length)
  - Debye length
- Collisional plasma: ideal MHD
  - MHD waves
  - Reduced MHD
- Weakly collisional plasma: non-ideal MHD
  - Ambipolar diffusion
  - Hall effect
- Collisionless plasma
  - hybrid-kinetics
  - gyro-kinetics

Plasma = collection of charged particles

$m_p \bullet q_p$ ← charge

↗ mass

← magnetic field

$$m_p \frac{dv_p}{dt} = q_p\left[E + \frac{v_p \times B}{c}\right] \qquad \frac{dx_p}{dt} = v_p$$

↑ electric field

Distribution function = probability of finding particle at certain point in phase space (x, v)

$$f(x, v, t) = \frac{\partial^2 N}{\partial x \, \partial v}$$

$$\int f \, d^3v = n(x) \quad \leftarrow \text{particle number density}$$

$$\int v f \, d^3v = n(x)\, u(x) \quad \leftarrow \text{particle momentum}$$

$\dots$

For a single particle,

$$f_1(x, v, t) = \delta(x - x_p) \cdot \delta(v - v_p)$$

For N particles

$$f_N(x, v, t) = \sum_{p=1}^{N} \delta(x - \underbrace{x_p}_{x_p(t)})\, \delta(v - \underbrace{v_p}_{v_p(t)})$$

How does f evolve with time?

$$\frac{\partial f_N}{\partial t} = \frac{\partial}{\partial t} \sum_{p=1}^{N} \delta(x - x_p(t))\, \delta(v - v_p(t)) =$$

$$= \sum_{p=1}^{N} \left[+ \frac{dx_p}{dt} \frac{d}{dx_p} \delta(x - x_p)\, \delta(v - v_p) + \frac{dv_p}{dt} \frac{d}{dv_p} \delta(x - x_p)\, \delta(v - v_p) \right] =$$

$$= -\sum_{p=1}^{N} \left\{ v_p \frac{\partial}{\partial x} \delta(x - x_p)\, \delta(v - v_p) + \frac{q_p}{m_p}\left(E + \frac{v_p \times B}{c}\right) \frac{\partial}{\partial v} \delta(x - x_p)\, \delta(v - v_p) \right\}$$

$\delta(v-v_p)$ is non-zero only for $v = v_p \Rightarrow$

$\Rightarrow \quad v_p \frac{\partial}{\partial x} \delta \rightarrow v \frac{\partial}{\partial x} \delta$

$$\frac{\partial f}{\partial t} = - v \frac{\partial}{\partial x}\left(\sum_p \delta(x-x_p)\,\delta(v-v_p)\right) - \frac{q}{m_p}\left(E + \frac{v \times B}{c}\right)\frac{\partial}{\partial v}\left(\delta(\ldots)\,\delta(\ldots)\right)$$

$$= - v \cdot \nabla f - \frac{q}{m_p}\left(E + \frac{v \times B}{c}\right)\frac{\partial}{\partial v} f$$

$\Downarrow$

f evolves according to

(Klimontovich)

$$\frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{q}{m}\left(E + \frac{v \times B}{c}\right)\frac{\partial f}{\partial v} = 0 \quad \text{– Kinetic equation}$$

E and B evolve according to Maxwell's equations

$$\nabla \cdot B = 0 \qquad \nabla \cdot E = 4\pi \rho_c$$

$$\underbrace{\nabla \times B = \frac{4\pi}{c} J_c + \frac{1}{c}\frac{\partial E}{\partial t}}_{\text{Ampère}} \qquad \underbrace{\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}}_{\text{Induction}}$$

charge density

$\rho_c \equiv \sum_s q_s n_s \leftarrow$ sum over species

current density

$J_c \equiv \sum_s q_s n_s u_s$

E and B in Klimontovich equation are „microscopic"

We typically want to have a smooth distribution function instead of delta function $\Rightarrow$ $f \rightarrow \langle f \rangle$, where $\langle\,\rangle$ is averaging over a small volume of phase space

$E = \langle E \rangle + \delta E, \qquad B = \langle B \rangle + \delta B$

$\downarrow$

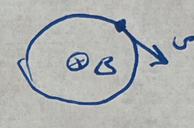
$$\left(\frac{\partial}{\partial t} + v \cdot \nabla + \frac{q}{m}\left(\langle E \rangle + \frac{v}{c} \times \langle B \rangle\right) \cdot \frac{\partial}{\partial v}\right) f = \overbrace{-\left\langle \frac{q}{m}\left(\delta E + \frac{v}{c} \times \delta B\right)\frac{\partial \delta f}{\partial v}\right\rangle}^{\text{collision integral}}$$

Particle in uniform E and B fields

$\frac{dx}{dt} = v \qquad m\frac{dv}{dt} = qE + q\,\frac{v\times B}{c}$

1) $B=0 \rightarrow v = \frac{q}{m}Et$ — linear acceleration

2) $E=0 \rightarrow \frac{dv}{dt} = \frac{q}{mc}\, v\times B$



$B = B_0\,\hat{e}_z$

$\left.\begin{aligned} \frac{dv_x}{dt} &= \frac{qB_0}{mc}\,v_y = \Omega\, v_y \\ \frac{dv_y}{dt} &= -\Omega\, v_x \end{aligned}\right\} \Rightarrow \begin{aligned} v &= v_0\, e^{i\Omega t} \\ x &= x_0\, e^{i\Omega t + i\pi/2} \\ x_0 &= v_0/\Omega \end{aligned}$

$\Omega = \frac{qB}{mc}$ = Larmor frequency, gyro frequency

$\rho_L = v/\Omega_i$ = Larmor radius, gyro-radius.

3) $E, B \neq 0$

$\frac{dv}{dt} = \frac{q}{m}\left(E + \frac{v\times B}{c}\right)$

if $E<B$, there is a frame, in which

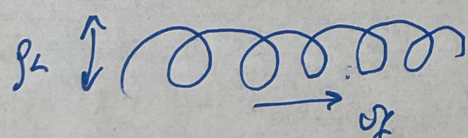
$E' = E + \frac{v_f\times B}{c} = 0$

$v_f = c\,\frac{E\times B}{B^2}$

in this frame $E'=0 \quad B'\neq 0 \Rightarrow$ particle rotates around B

$\Downarrow$

motion could be described as superposition of rotation and uniform motion



Invariants of particle motion

Hamiltonian of a particle in electromagnetic field

$$\mathcal{H} = \frac{1}{2m}\left(p - eA\right)^2 + e\varphi$$

$\nabla \times A = B$ $\qquad$ $E = -\nabla\varphi - \frac{\partial A}{\partial t}$

Poincaré invariant (see, e.g. Goldstein 1980)

$$P = \oint p \cdot \frac{dq}{d\lambda}\, d\lambda \approx \text{const}$$

$$\frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial q} \qquad \frac{dq}{dt} = \frac{\partial \mathcal{H}}{\partial p}$$

p, q – canonical coordinates and momenta

$$\frac{dP}{dt} = \oint \frac{d}{dt}\left(p\frac{dq}{d\lambda}\right) d\lambda = \oint\left\{-\frac{\partial \mathcal{H}}{\partial q}\frac{dq}{d\lambda} + p\frac{d}{d\lambda}\left(\frac{\partial \mathcal{H}}{\partial p}\right)\right\} d\lambda =$$

$$= \underbrace{\oint\left\{-\frac{\partial \mathcal{H}}{\partial q}\frac{dq}{d\lambda} - \frac{\partial \mathcal{H}}{\partial p}\frac{dp}{d\lambda}\right\} d\lambda}_{-\frac{d\mathcal{H}}{d\lambda}d\lambda \text{ integrates to zero}} + \underbrace{p\frac{\partial \mathcal{H}}{\partial p}\Big|_{\lambda_0}^{\lambda_0}}_{\text{"0" because the orbit is closed}}$$

$$\Downarrow$$

$$\frac{dP}{dt} = 0$$

For an ion in uniform field,

$p = m(v - eA/mc) \rightarrow$

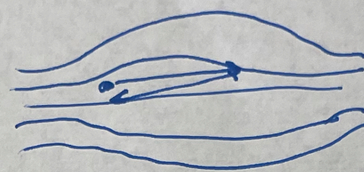
$$\rightarrow \oint p \frac{dq}{d\lambda} d\lambda = \int m v_\perp \, dq_\perp - \frac{e}{c}\int A\, dq_\perp = 2\pi m v_\perp \rho -$$

$$- \frac{e}{c} B (\pi \rho^2) \propto \frac{v_\perp^2}{B} \qquad \mu = \frac{mv^2}{B} = \text{const}$$

„magnetic moment"

Second adiabatic invariant

$$J = \oint p_\parallel \, dl$$



$$\mathcal{E} = \frac{m v_\parallel^2}{2} + \frac{m v_\perp^2}{2} = \text{const}$$

$$\mu = \frac{m v_\perp^2}{2eB} = \text{const}$$

$$\Downarrow$$

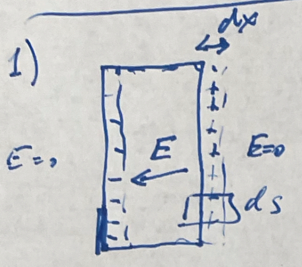
$$v_\perp^2 = \frac{2eB}{m}\mu$$

$$v_\parallel^2 = \frac{2\mathcal{E}}{m} - \frac{2eB}{m}\mu = \frac{2}{m}\left(\mathcal{E} - eB\mu\right)$$

$$J = \oint dl \sqrt{\frac{2}{m}} \sqrt{\mathcal{E} - eB\mu} \qquad B = B(l)$$

$v_\parallel = 0$ when $B(l) = \dfrac{\mathcal{E}}{e\mu}$

Basic collective plasma processes

1) 

$n, q, m.$

charge per unit area
$$\sigma = q n \cdot dx$$

$$E\,dS = 4\pi\sigma\,dS \Rightarrow E = 4\pi\sigma$$

restoring force per unit area
$$df = \sigma\,dS \cdot E = 4\pi\sigma^2\,dS$$

dm = mass per unit area $= m\,n\,dx$

$$dm\,\ddot{x} = -4\pi\sigma^2\,dS = -4\pi x^2 n^2 q^2\,dS$$

$$dS\,m\,n\,x\,\ddot{x} = -4\pi x^2 n^2 q^2\,dS$$

$$\Downarrow$$

$$\ddot{x} = -x \cdot \frac{4\pi n q^2}{m} = -\omega_p^2 x$$

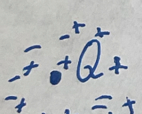
plasma oscillations with frequency

$$\omega_p = \sqrt{\frac{4\pi n e^2}{m_e}}$$

"plasma frequency"

"Langmuir waves"

2) Place a charge in quasi-neutral plasma



charged particles form a Boltzmann distribution

$$n_+ \simeq n_0 e^{-\frac{e\varphi}{T}} \approx n_0\left(1 - \frac{e\varphi}{T}\right)$$

$$n_- \simeq n_0 e^{\frac{e\varphi}{T}} \approx n_0\left(1 + \frac{e\varphi}{T}\right)$$

$$\rho_c = e n_+ - e n_- \approx -2 n_0 \frac{e^2\varphi}{T}$$

$$\nabla\cdot E = -\nabla^2\varphi = 4\pi\rho_c = 4\pi Q\,\delta(r) - \underbrace{\sum_s 4\pi n_s \frac{q_s^2}{T_s}}_{\text{for arbitrary number of species}} \cdot \varphi$$

$$\lambda_D^{-2} \equiv \sum_s 4\pi n_s \frac{q_s^2}{T_s} = \sum_s \lambda_{Ds}^{-2}$$

$$\lambda_{Ds} \equiv \frac{\sqrt{T_s/m_s}}{\omega_{ps}} = V_{ts}/\omega_{ps}$$

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$$-\nabla^2\varphi = 4\pi Q\,\delta(r) - \lambda_D^{-2}\varphi$$

$$\varphi = \frac{Q}{r} f(r) = \varphi_0 + \frac{Q}{r}(f(r)-1)$$

$$-\nabla^2\varphi = \underbrace{-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi_0}{\partial r}\right)}_{4\pi Q\,\delta(r)} - \frac{1}{r^2}\frac{\partial}{\partial}\left(r^2\frac{\partial}{\partial r}\left(\frac{Q}{r}(f-1)\right)\right)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left\{-\frac{Q}{r^2}(f-1)+\frac{Q}{r}f'\right\}\right] = \lambda_D^{-2}\frac{Q}{r}f$$

$$\frac{\partial}{\partial r}\left[-f+1+rf'\right] = \frac{r^2}{\lambda_D^2}\frac{1}{r}f$$

$$-f'+f'+rf'' = r^2/\lambda_D^2\,\frac{1}{r}f$$

$$\Downarrow$$

$$f'' = \frac{1}{\lambda_D^2}f \Rightarrow f = e^{-r/\lambda_D}$$

$$\Downarrow$$

$$\varphi = \frac{Q}{r}\cdot e^{-r/\lambda_D}$$

screening of charge at characteristic scale λ_D

$\Downarrow$

if $r \gg \lambda_D$ plasma is quasi-neutral

Scales in plasma

Larmor radius $\rho_s = v/\Omega_s$ $\Omega_s = \frac{q_s B}{m_s c}$

Skin-depth $d_s = c/\omega_{ps}$ $\omega_{ps} = \sqrt{\frac{4\pi n_s q_s^2}{m_s}}$

Debye length $\lambda_{Ds} = v_{ths}/\omega_{ps}$ $v_{ths} = \sqrt{T_s/m_s}$

λ_{mfp} – mean-free-path

L – typical scale

Dimensionless parameters

$\frac{L}{\lambda_{mfp}}$ – could be large or small

$$\frac{\rho_s}{d_s} = \frac{v_{th}/\Omega_s}{c/\omega_p} = \sqrt{\frac{T}{m}}\left(\frac{qB}{mc^2}\right)^{-1}\sqrt{\frac{4\pi n q^2}{m}} =$$

$$= \sqrt{\frac{4\pi T n}{B^2}} = \sqrt{\frac{1}{2}\left(\frac{8\pi n T}{B^2}\right)}$$

$$\frac{\rho_s}{d_s} = \sqrt{\beta_s/2} \quad \text{plasma } \beta \qquad \beta = \frac{\text{thermal } \cancel{\text{magnetic}} \text{ pressure}}{\cancel{\text{thermal}} \text{ magnetic pressure}}$$

$\frac{L}{\rho_s}$ – typically $\gg 1$ => plasma is magnetized

$\frac{\lambda_{Ds}}{d_s} = \frac{v_{ths}}{c}$ – small in non-relativistic plasmas =>

=> plasma is typically quasi-neutral

$$\frac{\partial f}{\partial t} + v\cdot\nabla f + \frac{e}{m}\left(E+\frac{v\times B}{c}\right)\frac{\partial f}{\partial v} = C[f]$$

$\sum_\alpha \int dv\, C[f] = 0$ ⇒ total number of particles is conserved during collision

$\sum_\alpha \int dv\, v\, C[f] = 0$ → total momentum is also conserved

$\sum_\alpha \int dv\, \frac{v^2}{2} C[f] = 0$ ⇒ total energy is conserved

$$\partial_t f + \nabla_x\cdot(v f) + \nabla_v\cdot\left(\frac{e}{m}\left(E+\frac{v\times B}{c}\right) f\right) = C[f]$$

$\int dv \ldots$

$\frac{\partial n}{\partial t} + \nabla_x(n u) = 0$ – continuity equation

$\int v\, dv \ldots$

$$\cancel{\frac{\partial n u}{\partial t} + (u\cdot\nabla_x)(n u)} + \frac{e}{m}\left(\cancel{nE + \ldots}\right) = 0$$

$$\int \cancel{v\cdot\nabla_v\left(f\frac{v\times B}{c}\right) dv} =$$

$$\cancel{= -\int f\frac{v\times B}{c}\, dv}$$

pressure tensor

$$\int v\frac{\partial f}{\partial t}\, dv = \frac{\partial}{\partial t}(n u)$$

$$\int v\cdot\nabla_x(f v)\, dv = \nabla_x\cdot\int f v v\, dv = \nabla_x\left(\rho u u + P\right)$$

$$\int v\cdot\nabla_v(E f)\, dv = -E n u$$

$$\int v\cdot\nabla_v\left(\frac{v\times B}{c} f\right) dv = -\frac{n u\times B}{c}$$

Sum over species

$$\sum_\alpha n_\alpha E = \rho_c E = 0 \quad \text{for } L \gg \lambda_D$$

$$\sum_\alpha n_\alpha \frac{v_\alpha \times B}{c} = \frac{J \times B}{c}$$

$$u = \frac{1}{\rho_m} \sum_\alpha m_\alpha n_\alpha v_\alpha$$

↖ center of mass velocity

$\Downarrow$

$$\frac{\partial \rho u}{\partial t} + \nabla \cdot (\rho u u + P) = \frac{J \times B}{c}$$

$\rho u \cdot \nabla u$

$P = pI + \Pi$ ← viscous stress

Ideal hydrodynamics

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0$$

$$\frac{\partial \rho u}{\partial t} + \rho u \cdot \nabla u = -\nabla p$$

1 characteristic velocity – sound speed

$$c_s^2 \approx \frac{p}{\rho}$$

$\delta u \gtrsim c_s$ – strong compressions, shocks

$\delta u \ll c_s$ – no shocks, sound waves equilibrate pressure

Ideal MHD

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0$$

$$\frac{\partial \rho u}{\partial t} + \rho (u \cdot \nabla) u = -\nabla p + \frac{J \times B}{c}$$

$$\nabla \times B = \frac{4\pi}{c} J$$ ← neglect displacement current

$$\frac{1}{c} \frac{\partial B}{\partial t} = -\nabla \times E \leftarrow E + \frac{u \times B}{c} = 0$$ – "frozen-in condition"

$$\frac{\partial B}{\partial t} = \nabla \times (u \times B)$$

new characteristic velocity associated with $J \times B$ term

$$v_A \simeq \frac{B_0}{\sqrt{4\pi \rho}}$$ Alfvén speed

1 continuity equation
3 momentum equation
3 induction equation + 1 constraint $\nabla \cdot B = 0$
1 energy equation

7 wave solution

1 entropy wave
2 Alfven waves
2 slow magnetosonic modes
2 fast magnetosonic modes

$\Big\}$ analogous to sound waves.

Non-ideal MHD

$$\frac{\partial B}{\partial t} = \nabla \times (u \times B)$$

what is u?

Consider a collisional plasma

$$m_n n_n (\partial_t + u_n \nabla) u_n = -\nabla p_n + R_{ni} + R_{ne}$$

$$m_i n_i (\partial_t + u_i \nabla) u_i = -\nabla p_i + R_{in} + R_{ie} + q_i n_i \left(E + \frac{u_i \times B}{c}\right)$$

$$m_e n_e (\partial_t + u_e \nabla) u_e = -\nabla p_e + R_{en} + R_{ei} - e n_e \left(E + \frac{u_e \times B}{c}\right)$$

ohmic dissipation

Hall effect

Ambipolar diffusion

$$\frac{\partial B}{\partial t} = \nabla \times \Big[(u_{\xi} - u_e) \times B + (u_e - u_i) \times B + (u_i - u_n) \times B + \underbrace{u_n \times B}_{\simeq\, u \times B} \Big]$$

$\uparrow$ center of mass

small

could be large

$$\frac{\mathrm{Ohm}}{u \times B} \sim \frac{1}{R_m} \equiv \frac{\eta}{v_A l_B} \sim \left(\frac{d_e}{l_B}\right)\left(\frac{d_e}{v_A \tau_{en}}\right)$$

$$\frac{\mathrm{Hall}}{u \times B} \sim \frac{|j/e n_e|}{|u_n|} \sim \underbrace{\left(\frac{d_i}{l_B}\right)}_{\ll 1} \underbrace{\left(\frac{\rho_m}{\rho_{ni}}\right)^{1/2}}_{\sim 1} \underbrace{\left(\frac{v_A}{u_n}\right)}_{\sim 1} \mathrm{a.f.}$$

$$\frac{\text{Ambipolar}}{\eta_0 B} \sim \left|\frac{R_{ni}\tau_{ni}}{\rho_n u_n}\right| \sim \left(\left|\frac{j\times B}{c}\right| \frac{\tau_{ni}}{\rho_n v_A} \frac{v_A}{u_n}\right)$$

Collisionless plasma

fully - kinetic

$$\frac{\partial f_i}{\partial t} + v\cdot\nabla f_i + \frac{q_i}{m_i}\left(E + \frac{v\times B}{c}\right)\frac{\partial f_i}{\partial v} = 0$$

$$\frac{\partial f_e}{\partial t} + v\cdot\nabla f_e + \frac{q_e}{m_e}\left(E + \frac{v\times B}{c}\right)\frac{\partial f_e}{\partial v} = 0$$

hybrid - kinetic : $\ell \sim d_i, \rho_i \gg d_e, \rho_e$

$m_e/m_i \ll 1$

Equation of motion for electrons gives E

$$\cancel{m_e \frac{du_e}{dt}} \approx 0 = E + \frac{u_e\times B}{c} - \nabla P_e$$

MHD waves

$$\frac{J\times B}{c} = \frac{(\nabla\times B)\times B}{4\pi}$$

$$\frac{\partial \rho u}{\partial t} = -\nabla\cdot\Big(\rho u u + p\underline{I} + \underbrace{\frac{B^2}{8\pi}\underline{I} - \frac{B^2}{4\pi}\hat{b}\hat{b}}\Big)$$

$$\frac{\partial B}{\partial t} = \nabla\times(u\times B)$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho u) = 0$$

$$k\cdot\delta B = 0 \Rightarrow \delta B_{\parallel} = 0$$

$$\frac{\partial e}{\partial t} + \nabla\cdot(e u) = -p\,\nabla\cdot u \longrightarrow \left(\frac{\partial}{\partial t} + u\cdot\nabla\right)s = 0$$

$$s = (\gamma-1)^{-1}\ln\left[P\rho^{-\gamma}\right]$$

linearization

$$B = B_0 + \delta B$$

$$u = u_0 + \delta u\ , \quad u_0 = 0$$

$$\rho = \rho_0 + \delta\rho$$

$$p = p_0 + \delta p$$

$$\delta A = A e^{-i\omega t + ikx}$$

$$\delta\rho = \rho_0\frac{k\cdot\delta u}{\omega}$$

$$-i\omega\,\delta\rho + i\rho_0 k\cdot\delta u = 0$$

$$i\omega\left[\delta p\,\rho^{-\gamma} - \gamma\,\delta\rho\, p\,\rho^{-\gamma-1}\right] = 0 \rightarrow \omega = 0 \quad \text{entropy wave}$$

$$\frac{\delta p}{p} = \gamma\frac{\delta\rho}{\rho}$$

$$-i\omega\rho_0\,\delta u = -ik\,\delta p + (ik\times\delta B)\frac{\times B_0}{4\pi}$$

$$-i\omega\,\delta B = ik\times(\delta u\times B_0)$$

$$-\omega\,\delta u = -k\,\underbrace{\gamma\frac{p_0}{\rho_0^2}\rho_0}_{c_s^2}\frac{k\cdot\delta u}{\omega} + \frac{[k\times\delta B]\times B_0}{4\pi\rho_0}$$

$$-i\omega\,\delta u = -k\,c_s^2\,\frac{k\cdot\delta u}{\omega} + \frac{[k\times\delta B]\times B_0}{4\pi\rho_0}$$

$k = k_\parallel$ if $\delta u = \delta u_\parallel$

$$-i\omega \delta B = i k_\parallel \times (\delta u \times B_0) = 0 \quad \Rightarrow \quad \delta B = 0$$

$$-\omega \delta u = -k \gamma \frac{p_0}{\rho_0} \rho_0 \frac{k_\parallel \delta u_\parallel}{\omega}$$

Sound waves $\omega = \pm k_\parallel c_s$

$\delta u = \delta u_\perp$

$$-i\omega \delta B_\perp = i \delta u_\perp k_\parallel B_0$$

$$\delta B_\perp = \delta u_\perp \frac{k_\parallel}{\omega} B_0$$

$$-\omega \delta u_\perp = \frac{\left(k_\parallel \times \delta u_\perp \frac{k_\parallel}{\omega} B_0\right) \times B_0}{4\pi \rho_0} =$$

$$= -\frac{k_\parallel^2 V_A^2}{\omega} \delta u_\perp \qquad V_A = \frac{B_0}{\sqrt{4\pi \rho}} \quad \text{Alfven waves}$$

$k = k_\parallel + k_\perp$

$$\left(\omega^2 - k_\parallel^2 V_A^2\right)\left[\omega^2 - k_\parallel^2 V_A^2 - k_\perp^2 V_A^2 \frac{\omega^2}{\omega^2 - k^2 c_s^2}\right] = 0$$

$$\omega^2 = \frac{k^2 (c_s^2 + V_A^2)}{2} \pm \sqrt{\frac{k^4 (c_s^2 + V_A^2)^2}{4} - k_\parallel^2 V_A^2 k^2 c_s^2}$$

fast and slow modes

Mean free path

$n \sigma \lambda_{mfp} = 1$

$\sigma \sim \pi b^2 \qquad \frac{q^2}{b} \sim T \rightarrow \lambda_{mfp} \propto T^2/n$

$$\tau_{ei} \sim \frac{3\sqrt{m_e} T_e^{3/2}}{4\sqrt{2\pi}\, n_e \Lambda_e e^4}$$

$$\tau_{ii} \sim \frac{3}{4\sqrt{\pi}} \frac{\sqrt{m} T_i^{3/2}}{n_i \Lambda_i e^4}$$

$$\lambda_{mfp,e} = \frac{3}{4\sqrt{\pi}} \frac{T_e^2}{n_e} \frac{1}{\Lambda_e e^4}$$

$$\lambda_{mfp,i} \approx \frac{3\sqrt{2}}{4\sqrt{\pi}} \frac{T_i^2}{n_i \Lambda_i e^4} \approx \lambda_{mfp,e}$$

$$\lambda_{mfp,i} \approx \lambda_{mfp,e} \simeq \lambda_{mfp}$$