Squeezed Light in GW Interferometry (Big Picture)

HSC: I
QM laser

Last time on Matías' lecture:

Two types of laser noise:

1) Shot Noise

- Fluctuations in the photon count hitting the ~~mirrors~~ optical elements (mirrors / beam splitter / dark port etc.)
- ~~Poisson Statistics~~ • Error in the laser phase shift
- ~~Error~~ • Poisson statistics

∴ reduced by increasing laser power P_{in}

2) Radiation ~~Reaction~~ Pressure Noise (RPN)

- Fluctuations in the radiation pressure acting on optical elements
- Error in the force amplitude

∴ reduced by decreasing laser power P_{in}

→ Truly quantum mechanical in nature (vacuum fluctuations)

→ That increasing P_{in} would inadvertently ↑ RPN ↓ shot, or vice versa, is really a "balancing act"

↓

"Heisenberg Uncertainty Principle"

$$S_n(f)\big|_{\text{quantum}} = S_n(f)\big|_{RRN} + S_n(f)\big|_{SN}$$

$$= \frac{1}{2} S_{SQL}(f)\left[\frac{1}{K(f)} + K(f)\right] \quad \text{— ①}$$

$$S_{SQL} \equiv \frac{1}{2\pi f L}\sqrt{\frac{8\hbar}{M}}$$ ("Standard Quantum Limit")

Long arms

If M is too large, gravitational interaction between mirror and suspension system generates large gravity gradient noise (fluctuations in density of air)

$$K(f) = \frac{8\omega_L P}{ML^2}\frac{1}{\omega^2 + (\omega^2 + \omega_p^2)},$$

$\omega \equiv 2\pi f$

$\omega_L = 2\pi f_L$ (laser)

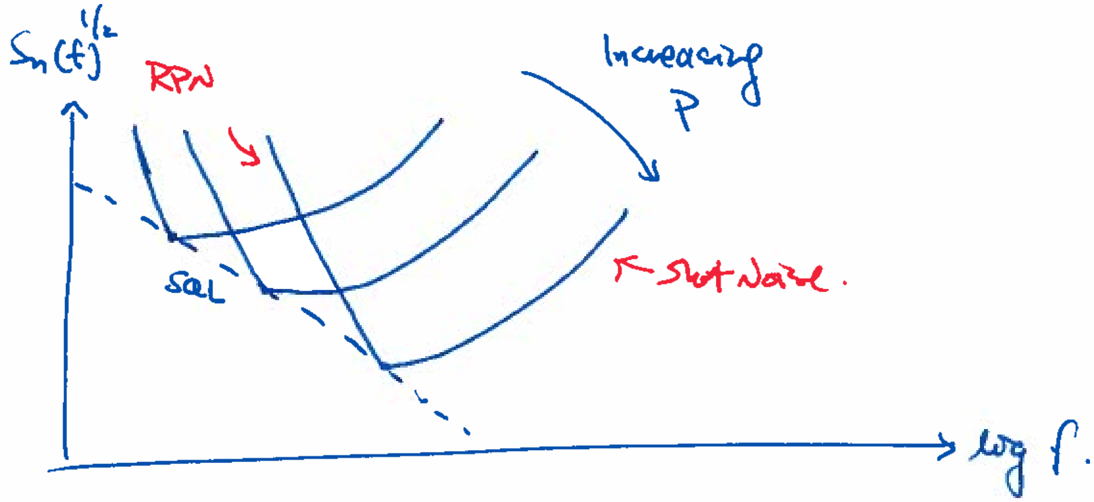
$\omega_p \sim \frac{c}{L}$ (inverse of light travel time)

"Optomechanical Coupling"

↓

Encodes interaction between laser and mirror

(Very important quantity for squeezing)



↑ P reduces shot Noise but increases Radiation Pressure & vice versa

∴ In a conventional interferometer, one cannot do better than this

Heuristically (to be discussed in more detail)

Uncertainty principle

$$\underline{\underline{\Delta N}}\,\underline{\Delta\phi} \geq 1.$$

RPN ↙ ↘ Shot Noise

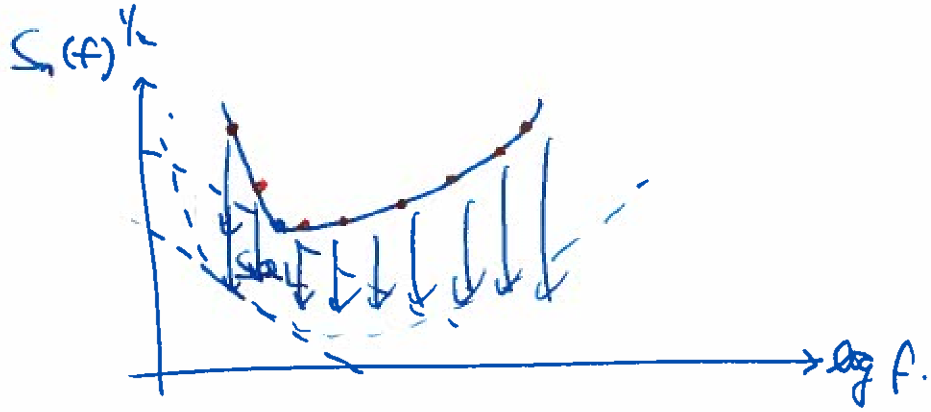
~~ΔN ↑ means~~

ΔN can be made very small, RPN ↓ but at the expense of increasing Δφ, SN ↑ & vice versa.

Squeezing is a method of significantly reducing the uncertainty of either ΔN or $\Delta\phi$, at the expense of increasing the uncertainty of the other (conjugate variable)

So how does squeezing really help ???

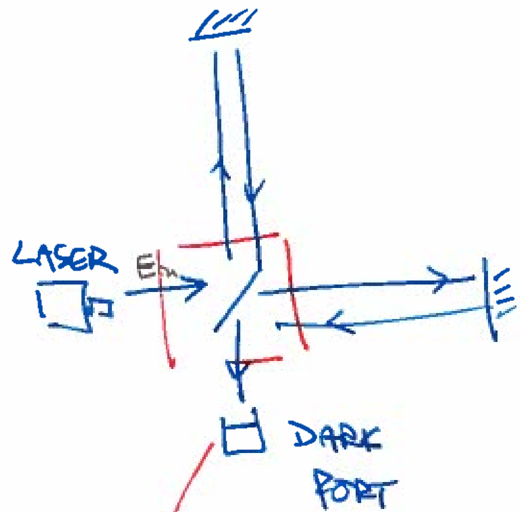
FREQUENCY-DEPENDENT SQUEEZING !



Squeeze $\Delta N(f)\Delta\phi(f) \geq 1$

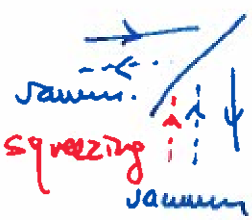
At each f, change the squeezing angle "θ" to minimize the uncertainty at that frequency range

→ The performance of laser quantum noise depends heavily on this squeezing angle, and we will see how the curve changes depending on choice of θ



Usually, we ~~[illegible]~~ think of the dark port simply as a readout port, where we do not inject any signal from the DP to the beamsplitter.

However, quantum mechanically, nothing $\Leftrightarrow$ vacuum, and vacuum is interesting!



fluctuations of EM field.

~~To achieve squeezing, we squeeze the vacuum state.~~

→ I will show how we can derive $S_n(f)|_{quantum}$ using quantum mechanics from the picture above

→ How by replacing the vacuum state with a squeezed vacuum, we can shape the formula for ①

Some Basics of Quantum Optics.

HSC: II
QM laser

In quantum mechanics, the Electromagnetic field is promoted to an operator and acts like a simple harmonic oscillator.

$$\hat{E}(t) = \mathcal{E}_0 \left[\hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{+i\omega t} \right]$$

where $\hat{a}$: annihilation operator
$\hat{a}^\dagger$: creation operator.

$\hat{a}|0\rangle = 0$, $\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$, $\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$|0\rangle$: vacuum , $|n\rangle$: n-particle state

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle$$

$[\hat{a}, \hat{a}^\dagger] = 1$, $[\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 1$

It is convenient to define the quadrature operators:

$$\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger), \quad \hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger)$$

such that

$$\boxed{\vec{E}(t) = 2\mathcal{E}_0 \left[\hat{X}_1 \cos\omega t + \hat{X}_2 \sin\omega t \right].}$$

called quadrature because these terms differ by $\pi/2$ in phase

- obviously $\hat{X}_1$ and $\hat{X}_2$ are Hermitian ∴ real observables.

Using ~~the~~ trigonometric identity, we can rewrite

" $\vec{E}(t) = 2\mathcal{E}_0 \sqrt{\hat{X}_1^2 + \hat{X}_2^2} \cos(\omega t + \beta), \quad \beta = \tan^{-1}\left(\frac{\hat{X}_2}{\hat{X}_1}\right)$ "

illegal strictly because $\hat{X}_1$ and $\hat{X}_2$ are operators, and I have not specified the states that $\vec{E}$ is acting upon.

Key Points: $\{\hat{X}_1, \hat{X}_2\} \Longleftrightarrow \{$Amplitude, phase$\}$

more convenient to work with

The commutation relations above imply

$$[\hat{X}_1, \hat{X}_2] = i/2$$

Using the identity $[\hat{A}, \hat{B}] = \hat{C}$, $\Delta A \Delta B \geq \frac{1}{2} |\langle \hat{C} \rangle|$ Heisenberg uncertainty relation

$$\Delta X_1 \Delta X_2 \geq \frac{1}{4}$$

How to visualize these abstract math?

① Consider acting $\vec{E}$ onto the vacuum state $|0\rangle$:

$$\langle 0|\hat{X}_1|0\rangle = \langle 0|\hat{X}_2|0\rangle = 0$$

$$\langle \Delta\hat{X}_1^2\rangle_0 = \langle 0|\hat{X}_1^2|0\rangle - (\langle 0|\hat{X}_1|0\rangle)^2 = \frac{1}{4}$$

$$\langle \Delta\hat{X}_2^2\rangle_0 = \frac{1}{4}$$

$$\Delta X_1 = \Delta X_2 = \frac{1}{2}$$

Maybe show once

$$= \langle 0|\, \cancel{aa} + \cancel{a^\dagger a^\dagger} + aa^\dagger + \cancel{a^\dagger a}|0\rangle \tfrac{1}{4}$$

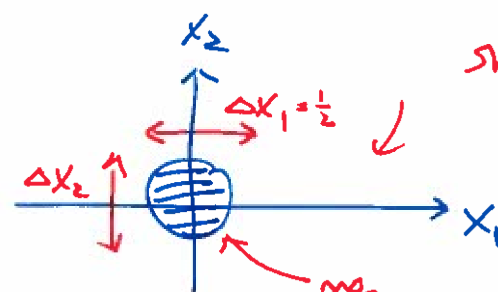
$$= \langle 0|\, 1 + \cancel{a^\dagger a}|0\rangle \times \tfrac{1}{4} = \tfrac{1}{4}$$

∴ the vacuum state minimizes the Heisenberg uncertainty relation

Draw the Phase Diagram:

Motivated by

"$\vec{E} = 2\xi_0\sqrt{\hat{X}_1^2 + \hat{X}_2^2}\,\cos(\omega t + \beta)$"



shaded region = area of uncertainty

mean = 0

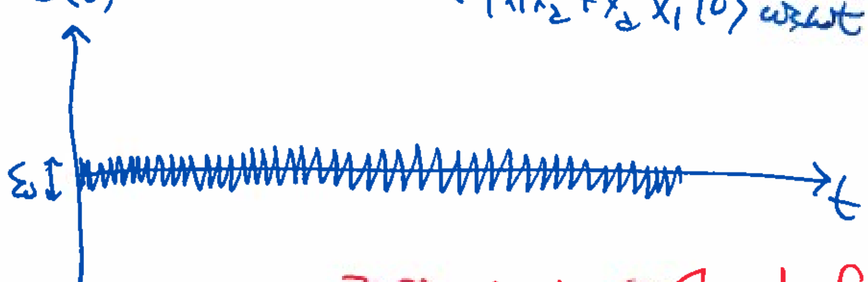
circle ⇒ equal uncertainty in both X_1 and X_2

To visualize E in the vacuum state:

$$\langle 0|\hat{E}|0\rangle = 0$$

$$\langle 0|\hat{E}^2|0\rangle = 4\xi_0^2\left(\langle 0|\hat{X}_1^2|0\rangle\cos^2\omega t + \langle 0|\hat{X}_2^2|0\rangle\sin^2\omega t\right) = \xi_0^2$$

$$+ \langle 0|\hat{X}_1\hat{X}_2 + \hat{X}_2\hat{X}_1|0\rangle\cos\omega t\,\sin\omega t$$



zero mean, constant vacuum fluctuation in time

② Consider the coherent state

(the most classical quantum state!)

→ this really is the laser light LIGO uses (see below)

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \quad \alpha = |\alpha|e^{i\phi}$$

non-Hermitian, ∴ α is imaginary

Solving for this eigenvalue equation, one can show that

$$|\alpha\rangle = \exp\left(-\tfrac{1}{2}|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

To get an intuitive idea for its meaning, compute

$$\langle\alpha|\hat{E}|\alpha\rangle = \langle\alpha|\,\mathcal{E}_0(\hat{a}e^{-i\omega t} + \hat{a}^{\dagger}e^{+i\omega t})|\alpha\rangle$$

$$= \mathcal{E}_0(\alpha e^{-i\omega t} + \alpha^* e^{+i\omega t})$$

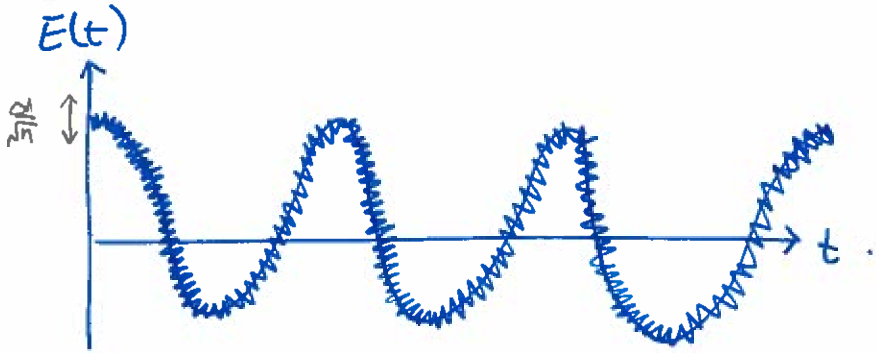
$$= 2\mathcal{E}_0|\alpha|\cos(\omega t - \phi)$$

which looks like a classical E field with amplitude $\propto \mathcal{E}_0|\alpha|$

$$\langle\alpha|\hat{E}^2|\alpha\rangle = \mathcal{E}_0^2\left(1 + 4|\alpha|^2\cos^2(\omega t - \phi)\right)$$

$$\langle\Delta\hat{E}\rangle_\alpha = \langle\alpha|\hat{E}^2|\alpha\rangle - \left(\langle\alpha|\hat{E}|\alpha\rangle\right)^2 = \mathcal{E}_0^2$$

which is identical to vacuum fluctuations.



✓ quantum fluctuations on top of a classical E field

Vacuum (as close to a classical field as is possible ~~for~~ any quantum state)

Projecting $|\alpha\rangle$ onto $|n\rangle$, and taking the square for probability,

$$P_n = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}$$

∴ Poisson distribution with mean $|\alpha|^2 \equiv \bar{n}$, where $\bar{n}$ is the average photon number

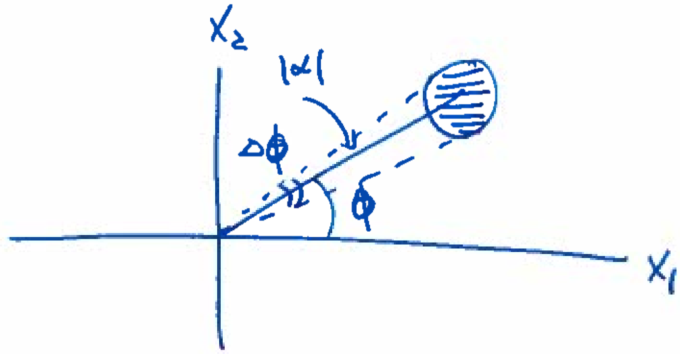
∴ $\Delta n = \bar{n}^{1/2}$

phase also fluctuates with a distribution (Gaussian in large $\bar{n}$ limit)

∴ From $\langle\alpha|\vec{E}|\alpha\rangle = 2\mathcal{E}_0|\alpha|\cos(\omega t-\phi) = 2\mathcal{E}_0\sqrt{\bar{n}}\cos(\omega t-\phi)$.

Amplitude fluctuates with Poisson distribution

(annotate in figure above)



- Displaced mean from centre because $\bar{n} = |\alpha|^2 > 0$
- Uncertainty area is the same as that of vacuum fluctuations

i.e. we show show that

$$(\Delta X_1)_\alpha = (\Delta X_2)_\alpha = \frac{1}{2}$$

$\therefore$ saturates the Heisenberg uncertainty relation.

Note: in large $|\alpha| = \bar{n}^{1/2}$ limit, $\Delta\phi \to 0$ (well-defined phase)

$\therefore$ Truly a classical limit in which both $\bar{n} \to \infty$, $\Delta\phi \to 0$

↗ Small $\bar{n}$, $\Delta n \propto \bar{n}^{1/2}$ is small & $\Delta\phi$ is large } radiation pressure noise ↓ shot noise ↑

↘ Large $\bar{n}$, $\Delta n \propto \bar{n}^{1/2}$ is large (Poisson fluctuation) & $\Delta\phi$ is small } radiation pressure ↑ shot noise ↓

Coherent states are truly the laser light used by the LIGO & Virgo detectors!

③ Squeezed States (Non-classical light)
(including squeezed vacuum and squeezed light)

In general, the Heisenberg uncertainty principle implies that

$$\Delta X_1 \Delta X_2 \geq \frac{1}{4}$$

A state is said to be squeezed if

$$\boxed{\Delta X_1 < \frac{1}{2} \quad \text{or} \quad \Delta X_2 < \frac{1}{2}}$$

cannot be satisfied simultaneously.

Mathematically, we achieve squeezing by applying the squeezing operator:

$$\hat{S}(\xi) = \exp\left[\frac{1}{2}\left(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2}\right)\right]$$

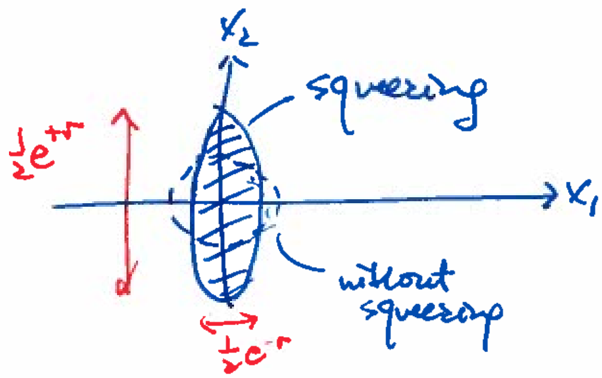
where $\xi = \underset{\substack{\downarrow \\ \text{squeeze factor}}}{r} e^{i\theta}$ $\theta \rightarrow$ squeeze angle, $r>0$

q-squeezed vacuum : $|0_s\rangle = \hat{S}(\xi)|0\rangle$

$$\langle 0_s|(\Delta\hat{X}_1)^2|0_s\rangle = \frac{1}{4}\left[\cosh^2 r + \sinh^2 r - 2\sinh r\cosh r\cos\theta\right]$$

$$\langle 0_s|(\Delta\hat{X}_2)^2|0_s\rangle = \frac{1}{4}\left[\cosh^2 r + \sinh^2 r + 2\sinh r\cosh r\cos\theta\right]$$

For $\theta = 0$, squeezing occurs along the X_1 quadrature.



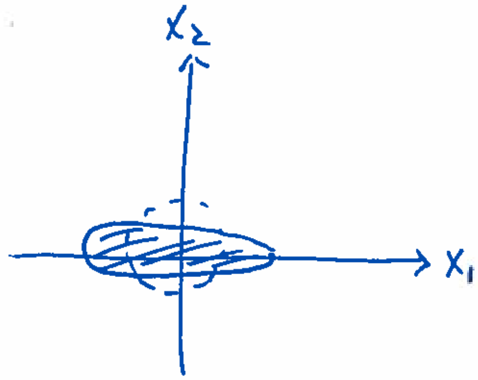
$\Delta X_1 \Delta X_2 = \frac{1}{4}$ in both cases so the area is preserved

$$\langle 0_s|\Delta\hat{X}_1|0_s\rangle = \frac{1}{2}e^{-r}, \quad \langle 0_s|\Delta\hat{X}_2|0_s\rangle = \frac{1}{2}e^{+r}$$

($r=0$ for vacuum)

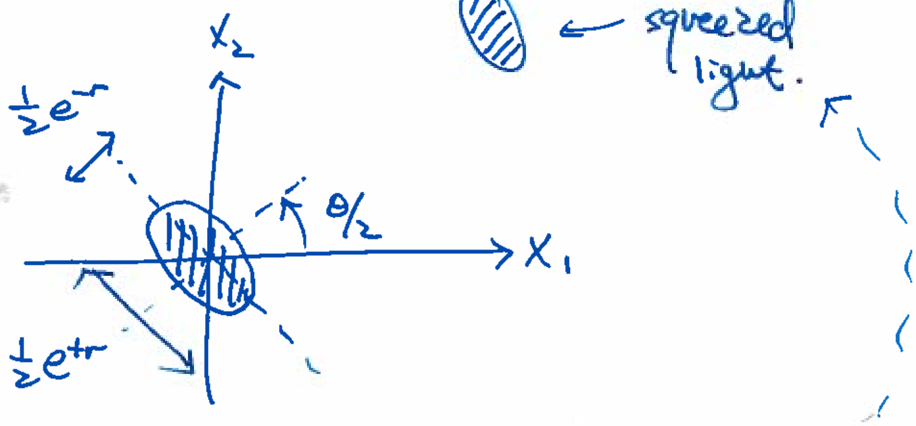
↳ Exponential factor compared to vacuum.

For $\theta = \pi/2$, squeezing occurs along the X_2 quadrature.



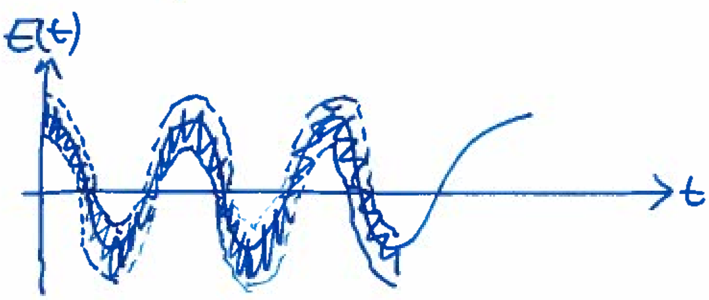
$$\langle 0_s|\Delta\hat{X}_1|0_s\rangle = \frac{1}{2}e^{+r}, \quad \langle 0_s|\Delta\hat{X}_2|0_s\rangle = \frac{1}{2}e^{-r}$$

For a general θ, squeezing would occur along a mixture direction in both X_1 and X_2.

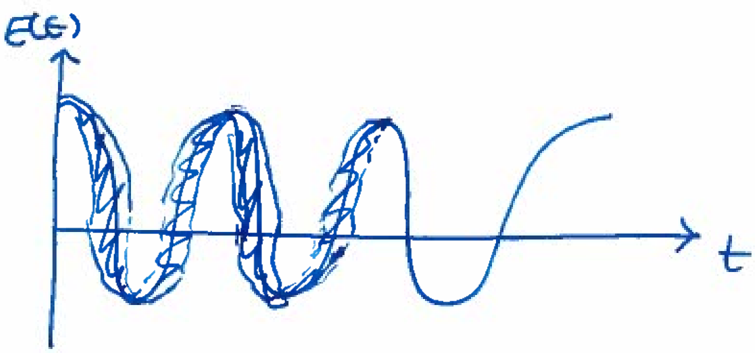


Similarly for squeezed lights, except that now ~~one~~ the ellipse is displaced with non-vanishing $\bar{n}$

How does squeezed light evolve in time?

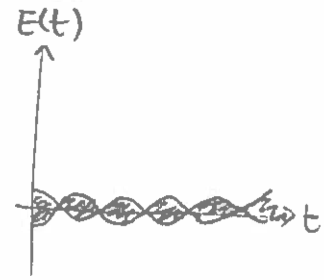


$\theta = 0$



$\theta = \pi/2$

Plot squeezed vacuum too.

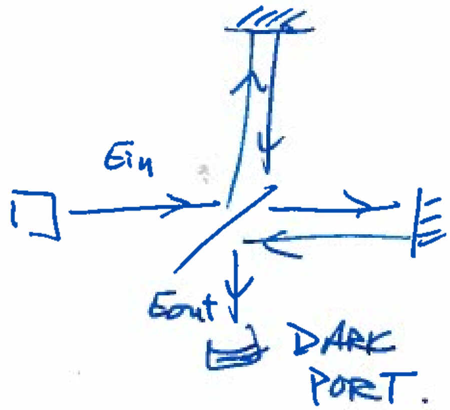


Key: Squeezing depends on the squeezing angle and is phase (time) dependent!

Deriving $S_n(f)$ for Squeezed Light

Vacuum &

HSC: IV
QM laser



laser power

laser frequency (Not GW freq!)

$$\hat{E}^{in} = 2\mathcal{E}_0 \left[(A^{in} + \hat{X}_1)\cos\omega_L t + \hat{X}_2 \sin\omega_L t \right]$$

$$\hat{E}^{out} = 2\mathcal{E}_0 \left[(B_1^{out} + \hat{Y}_1)\cos\omega_L t + (B_2^{out} + \hat{Y}_2)\sin\omega_L t \right]$$

$$\hat{X}_1, \hat{X}_2 = \int_{-\infty}^{\infty} df \; X_{1,2}(f)\, e^{-2i\pi f t}$$

→ creation & annihilation / quadrature operators for generating GW frequency

Matching reflection coefficients etc, we obtain the input/output relations [Kimble et al (2001)]

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = T \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + t \frac{h}{\sqrt{S_n(f)}}$$

$$T = e^{2i\beta(f)} \begin{pmatrix} 1 & 0 \\ -K(f) & 1 \end{pmatrix}, \quad t = \frac{1}{2} e^{i\beta(f)} \begin{pmatrix} 0 \\ \sqrt{2K(f)} \end{pmatrix}$$

Ponderomotive squeezing

off-diagonal part (quantum fluctuations in light acting on mirrors)

$$\beta = \tan^{-1}\left(\frac{2\pi f}{\omega_p}\right)$$

Recall

$$K(f) = \frac{8\omega_L P}{ML^2} \times \frac{1}{\omega^2(\omega^2+\omega_p^2)}$$

$$\hat{Y}_1 = e^{2i\beta}\hat{X}_1 \quad \text{(trivial)}$$

$$\underline{\underline{\hat{Y}_2}} = e^{2i\beta}\left(\underline{\underline{-K\hat{X}_1}} + \hat{X}_2\right) + e^{i\beta}\frac{\hat{h}\sqrt{2K}}{2\sqrt{S_{SQL}(f)}}$$

As before.

output

fluctuation noise

To derive the laser noise, it is equivalent to say that $\hat{h}$ is not from GW but from noise, $\hat{h} \to \hat{h}_n$, $\hat{Y}_2 = 2(-K\hat{X}_1 + \hat{Y}_2)$

(Kimble 2001)

$$\hat{h}_n = +\frac{2\sqrt{S_{SQL}}}{\sqrt{2K}}e^{i\beta}\left(-K\hat{X}_1 + \hat{X}_2\right)$$

Since $\langle \hat{h}_n^+ \hat{h}_n \rangle \equiv S_n(f)$ Definition of noise curve

For a quantum vacuum in the dark port,

$$\langle 0|\hat{h}_n^+\hat{h}_n|0\rangle = \frac{4S_{SQL}}{(2K)}\langle 0|\left(-K\hat{X}_1 + \hat{X}_2\right)^2|0\rangle$$

$$= \frac{4S_{SQL}}{(2K)}\left[K^2\underbrace{\langle 0|\hat{X}_1^2|0\rangle}_{=\frac{1}{4}} + \underbrace{\langle 0|\hat{X}_2^2|0\rangle}_{=\frac{1}{4}} + \text{mixing terms that vanish}\right]$$

$$= \frac{S_{SQL}}{2}\left[K + \frac{1}{K}\right]$$

(as before)

if we inject a squeezed vacuum at the dark port,

$$|0_s\rangle = S(r,\theta)|0\rangle$$

The noise curve

$$S_n(f) = \langle 0_s| h_n^\dagger h_n |0_s\rangle = \langle 0| S^\dagger(r,\theta)\, h_n^\dagger h_n\, S(r,\theta)|0\rangle$$

free to choose squeezing angle

$$= \frac{S_{SQL}}{2}\left(\frac{1}{K}+K\right)\left[\cosh 2r - \cos\left(2(\theta+\Phi)\right)\sinh 2r\right]$$

where $\Phi(f) = \cot^{-1} K(f)$

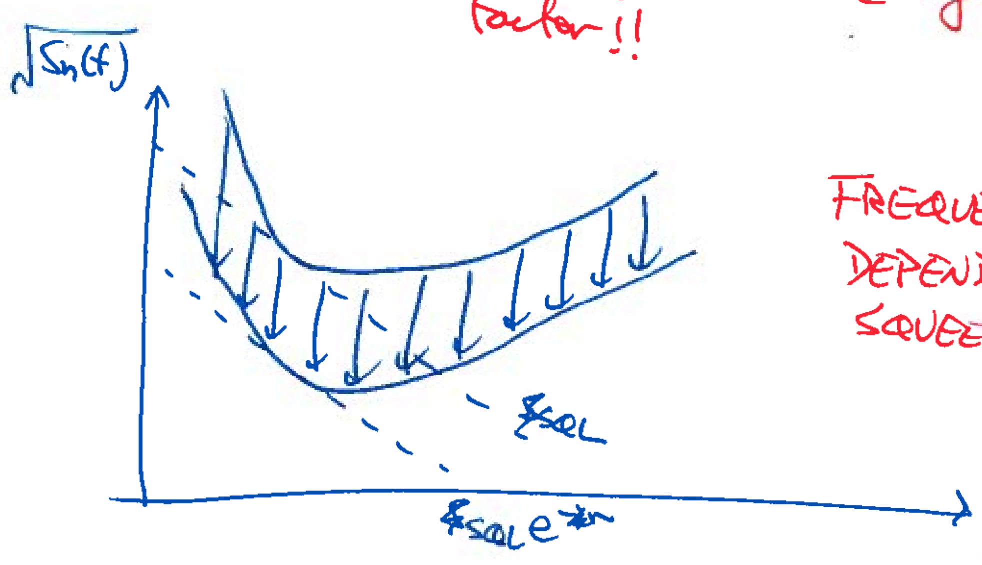
frequency dependent

1) The noise curve is minimized if we choose

$$\theta(f) = -\Phi(f) = -\cot^{-1} K(f)$$

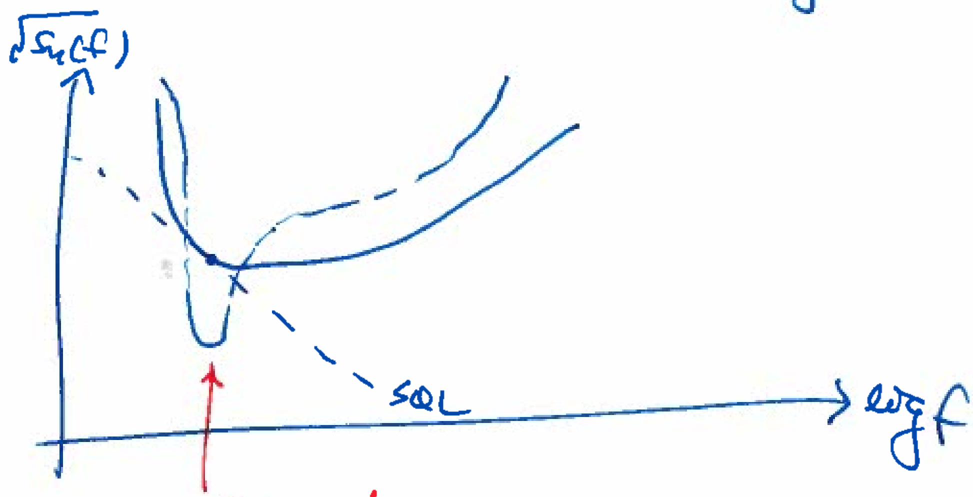
$$S_n(f) = \frac{S_{SQL}}{2}\left(\frac{1}{K}+K\right) e^{-2r}.$$

effectively reduces $\frac{S_{SQL}}{2}$ by an exponential factor!!



FREQUENCY DEPENDENT SQUEEZING!

2) If we choose θ to be a constant, say $\theta = \pi/4$.



→ beat quantum noise at a specific frequency

→ but noise is larger at smaller & larger frequencies

3) For $\theta = \pi/2$,

$$S_n = \frac{S_{SQL}}{2}\left(\frac{1}{Ke^{2N}} + Ke^{2N}\right)$$

$\therefore$ Produces same noise curve as vacuum, except that we can achieve the same curve with less input power ($K \propto P$)