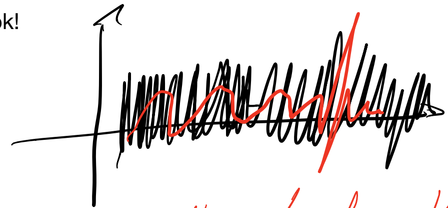
Largely based on Jonathan Gair's Lecture notes and Maggiore's book!

{Signal processing}

$$s(t) = h(t) + n(t)$$

signal noise



How to find the signal?

Noise $n(t)$ evolves according to probabilistic process.

$$p_N((n_N,t_N); \ldots; (n_1,t_1))\, dn_N \ldots dn_1$$

Stationarity on noise: Only depends on time difference

$$p_N((n_N, t_N+\tau), \ldots (n_1, t_1+\tau)) = p_N((n_N,t_N); \ldots; (n_1,t_1))$$

$\tau - \Delta t$

Can model noise as Gaussian

$$p \ldots = A \exp\left[-\frac{1}{2} \sum_j \sum_k \alpha_{jk} (y_j - \bar{y})(y_k - \bar{y})\right]$$

$y = y(t_i, n_i)$

Zero noise: $\int_{-T/2}^{T/2} n(t)\, dt = 0$

How about fluctuations: $\tilde{P} = \int_{-T/2}^{T/2} |n(t)|^2\, dt$

Power

$\sim$ const $\Rightarrow$ $\tilde{P} \sim$ const $\times T$ linear

$$\Rightarrow \boxed{P_u = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |u(t)|^2 dt}$$

Remember $\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} \hat{f}(f) df$ Parseval.

Noise is Real $\Rightarrow$ $\int_{-\infty}^{\infty} |\hat{u}(t)| dt = 2 \int_{0}^{\infty} |\hat{u}_T(f)| df$

$\hat{u} \times$ $-\frac{T}{2}$ $\frac{T}{2}$

$$\Rightarrow P_u = \lim_{T\to\infty} \frac{2}{T} \int_{0}^{\infty} |\hat{u}_T(f)|^2 df$$

T only f

define $= \int_0^\infty \underbrace{\lim_{T\to\infty} \frac{2}{T} |\hat{u}_T(f)|^2}_{S_u}\, df$

$\underline{S_u}$ $\underline{\text{spectral density}}$

$$= \int_0^\infty S_u(f) df$$

From the definition: Variation around Δt

$$\Delta u^2 = \lim_{N\to\infty} \frac{2}{N} \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} \left| \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} u(t)\, e^{2\pi i f t} dt \right|^2 = \frac{S_u(f)}{\Delta t}$$

$$\simeq S_u(f)\, \Delta f$$

$$\Rightarrow \boxed{\Delta u_{rms} = \sqrt{S_u(f)\,\Delta f}}$$

Wiener - Khinchine

Autocorrelation of u is related to the spectral density

$$\boxed{\langle u^*(t)\, u(t+\tau)\rangle = \int S_u(f)\, e^{2\pi i f \tau}\, df}$$

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} u^*(t)\, u(t+\tau)\, dt$$

Fourier transform

$$\boxed{\langle \hat{u}^*(f)\, \hat{u}(f')\rangle = S_u(f)\, \delta(f-f')}$$

Different types of noise

$S_u(f) = \text{const}$ white

$S_u(f) = \frac{1}{f}$ flicker

$S_u(f) = \frac{1}{f^2}$ random walk

Burst

$$\bar{P}_h = \frac{1}{\Delta t} \int_0^{\Delta t} |h(t)|^2\, dt = h_c^2$$

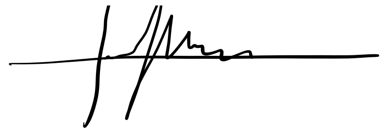
characteristic amplitude

bandwidth Δf

duration Δt

frequency f

From above noise power

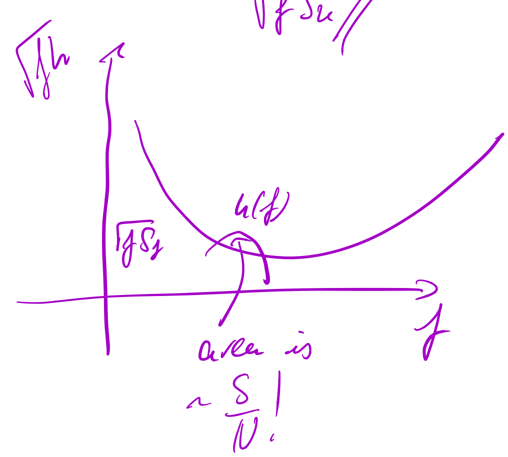


$|n| \sim \Delta f \, S_n(f)$

$$\Rightarrow \left(\frac{S}{N}\right)^2 = \frac{\overline{P_h}}{\Delta f \, S_n(f)} = \frac{h_c^2}{\Delta f \, S_n(f)} = \frac{h_c^2}{f \, S_n(f)}$$

broad band burst $\Delta f \sim f$

For a burst we should plot $h / \sqrt{f S_n}$



Alternative (LISA band)

continuous GW. Frequency largely constant

$$h(t) = h_0 \, e^{2\pi i f_0 t}$$

$$\Rightarrow P_h = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |h(t)|^2 \, dt = \frac{1}{2} h_0^2$$

If f_0 constant then error $\boxed{\Delta f \sim \frac{1}{T}}$

$$\Rightarrow |n|^2 \sim \Delta f \, S_n \sim \frac{S_n}{T}$$

So noise here goes like $\sqrt{\frac{S_n}{T}}$

observation time

<u>What about position dependent effects?</u>

$$\langle S_h(f)\rangle \simeq 5\, S_f(f) \qquad \text{LIGO}$$

sky position averaged

An inspiral of compact binaries has finite time and energy $\rightarrow \frac{1}{\sqrt{T}}\tilde{h}(f) \rightarrow 0 \quad T \rightarrow \infty$

Need to <u>window</u> the above expression

$$w(t) = \int_{-\infty}^{\infty} K(t-t')\, s(t')\, dt'$$

windowing kernel

$$\left(\frac{S}{N}\right)(t) = \frac{\int K(t-t')\, h(t')\, dt}{\sqrt{\langle \left| \int K(t-t')\, n(t')\, dt' \right|^2 \rangle}}$$

windowed noise

Remember
$$\boxed{\begin{aligned} & s = h + n \\ & \langle s\rangle_f = \langle h\rangle_f + \frac{\langle h_f\rangle}{\left(\frac{S}{N}\right)} \end{aligned}}$$

want to maximize this

In fourier space $\langle n(t)\, n(t-t')\rangle = \hat{n}^*\hat{n}$

$$\Rightarrow \frac{S}{N} = \frac{\int \hat{K}(f)\hat{h}(f)\,df}{\sqrt{\int |K(f)|^2 S_n(f)\,df}}$$

$\int |n^2| \sim \int S_n$

Define optimal filter

$$(h_1 | h_2) = 2\int_0^\infty \frac{\hat{h}_1^* h_2 + \hat{h}_1 \hat{h}_2^*}{S_n(f)}\,df$$

$$\Rightarrow \frac{S}{N} = \frac{(S_n K | h)}{\sqrt{(S_n K | S_n K)}}$$

maximize Wiener filter $\boxed{\hat{K} \propto \frac{\hat{h}_{template}}{S_n}}$

if $h = h_{template}$

$$\Rightarrow \left(\frac{S}{N}\right)[h] = \sqrt{(h|h)}$$

contains S_n!!

standard signal to NOISE

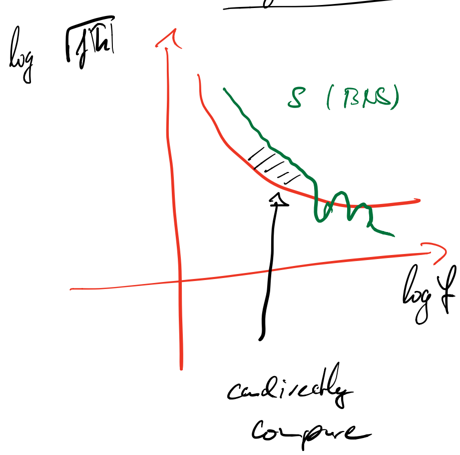
Practical observation

$$\left(\frac{S}{N}\right)^2 = 4\int_0^\infty \frac{f^2 |\hat{h}(f)|^2}{f S_n(f)}\frac{df}{f}$$

$$= 4\int_0^\infty \frac{f^2 |\hat{h}(f)|^2}{f\, S_n(f)}\, d\log f$$

compare log curves

$$\log\left(\frac{f^2 \hat{h}(f)^2}{f S_n(f)}\right) = 4\log\left(\sqrt{f}\hat{h}\right) - 2\log\left(\sqrt{f S_n}\right)$$



Area in log log plot $\propto \frac{S}{N}$

Parameter estimation

template will depend on $\vartheta_1, \ldots, \vartheta_D$

$$R \sim \frac{\hat{h}(f, \vartheta_\mu)}{S_n(f)}$$

From Wiener-Khinchine

$$\rightarrow \langle n^* n \rangle = \frac{1}{2} S_n(f)\, \delta(f-f')$$

$$\Rightarrow \quad p(n_0) = \mathcal{N} e^{-\frac{1}{2}\int_{-\infty}^{\infty} df \frac{|n_0(f)|^2}{\frac{1}{2} S_n(f)}}$$

gaussian noise

$$= \mathcal{N} e^{-\frac{(n_0|n_0)}{2}}$$

$\Rightarrow$ assume $\quad s(t) = h(t, \vartheta_\mu) + n_0(t)$

How likely is the signal?

$$p(n_0) = \mathcal{N} e^{-\frac{(s-h|s-h)}{2}} =: \Lambda(s, \Theta_t)$$

$$\Rightarrow \Lambda = \mathcal{N} e^{(h|s) - \frac{1}{2}(s|s)} e^{-\frac{1}{2}(h|h)}$$

prior probability p^0

$$\boxed{\Lambda_s = \tilde{\mathcal{N}} e^{(h|s) - \frac{1}{2}(h|h)}}$$

$\rightarrow$ Bayes $\quad P(\vartheta|s) = P(\vartheta_\mu)\, P(s|\vartheta)$

$$= P(\vartheta_\mu)\, \Lambda_s(s|\vartheta)$$

Need to fit ϑ_μ parameters:

For flat prior just use maximum likelyhood estimator

$$\log p \approx \log \Lambda = (h|s) - \frac{1}{2}(h|h) + \text{const}$$

extremize $\log p$
$d \log p = 0 \Rightarrow$ $\boxed{(\partial_i h \mid s) - (\partial_i h | h) = 0}$

template

This will find *some* local maximum

<u>Bayes estimator</u>

$$\vartheta^i_B(s) = \int d\vartheta\, \vartheta^i\, p(\vartheta|s)$$

← bayesian estimate for "optimal" error

$$\Sigma^{ij}_B = \int d\vartheta\, [\vartheta^i - \vartheta^i_B][\vartheta^j - \vartheta^j_B]\, p(\vartheta|s)$$

mean square deviations.

Assume that we have a good guess $\vartheta^i_B = \vartheta^i - \Delta\vartheta_i$

$\Delta\vartheta^i$ small

$$\Rightarrow p(\vartheta|s) = \mathcal{N} e^{-\frac{1}{2}\Gamma_{ij}\Delta\vartheta^i\Delta\vartheta^j}$$

taylor expand in $\Delta\vartheta^i$

$$\Gamma_{ij} = (\partial_i\partial_j h | \underbrace{h - s}_{-n}) + (\partial_i h | \partial_j h)$$

for high S/N $\rightarrow$ $\Gamma_{ij} \approx (\partial_i h | \partial_j h)$

$$\langle \Delta\vartheta_i \Delta\vartheta_j \rangle \simeq (\Gamma^{-1})^{ij}$$

property of the template!

How to beat GW signals of inspiralling binaries?

the simple template

$$h(t) = A_+ \, (\pi f_{GW})^{2/3} \cos(\Phi(f_{GW}) + \Phi_0)$$
$$+ A_\times \, (\pi f_{GW})^{2/3} \sin(\Phi(f_{GW}) + \Phi_0)$$

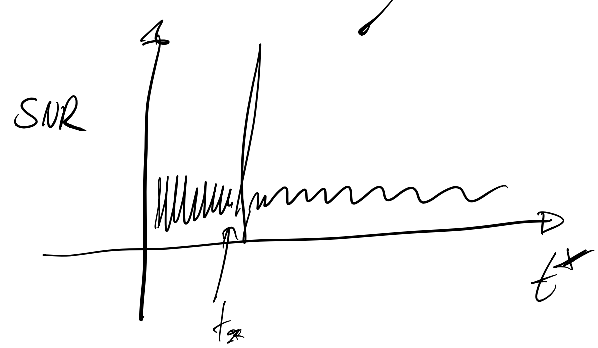
need to fix this

$$A_+ = \frac{4}{r} (\mathcal{M}_c)^{5/3} \, F_+(\theta, \phi) \, \frac{1 + \cos^2 \iota}{2}$$
$$A_\times = \frac{4}{r} \mathcal{M}_c^{5/3} \, F_\times(\theta, \phi) \, \cos \iota$$

Determining temporal offset

just appears as a phase in fourier space

$$(h(\theta, t_*) | s) \underset{\text{real}}{=} 4 \mathrm{Re} \int_0^\infty df \, \frac{\hat{h}^*(f, \theta) \, \hat{s}(f)}{S_n(f)} \, e^{i 2\pi f t_*}$$



How to fix Φ_0

$$h(t) = h_c(t) \cos\varphi + h_s(t) \sin\varphi \qquad | \text{real space}$$

Let's use the maximum likelihood estimator to get $\underline{A}$

Assume $h = A h_A(t, \vartheta_m)$

$$\Rightarrow \log \Lambda(s|A, \vartheta_m) = A(h_A|s) - \frac{A^2}{2}(h_A|h_A)$$

$$\frac{d \log \Lambda}{dA} = 0 \Rightarrow \boxed{A = \frac{(h_A|s)}{(h_A|h_A)}}$$ fixes the constant

$$\Rightarrow \boxed{\log \Lambda(s, \vartheta_u) = \frac{1}{2}\frac{(h_A|s)^2}{(h_A|h_A)}}$$

$$\Rightarrow 2 \log \Lambda = \frac{(h|s)^2}{(h|h)} = \frac{[(h_c|s) + (h_s|s)\tan\varphi]^2}{(h_c|h_c) + (h_s|h_s)\tan^2\varphi + 2(h_c|h_s)\tan\varphi}$$

Can introduce $\Phi_p, \Phi_c \Rightarrow$

$$h_p = h_c \cos\varphi_p + h_s \sin\varphi_p$$
$$h_q = h_c \cos\varphi_q + h_s \sin\varphi_q$$

$$\Rightarrow (h_p|h_q) = 0$$

$$\Rightarrow 2 \log \Lambda = \frac{(h_p|s)^2}{(h_p|h_p)} + \frac{(h_q|s)^2}{(h_q|h_q)}$$

can maximize these

On the detectability of sources:

$$\hat{h}(f) = \sqrt{\frac{5}{6}}\,\frac{1}{2\pi^{2/3}}\,\frac{\mathcal{M}_c^{5/6}}{D}\,f^{-7/6}\,e^{i\psi}\,Q(\theta,\phi,\iota)$$

orientation

Using the most optimal matched filter

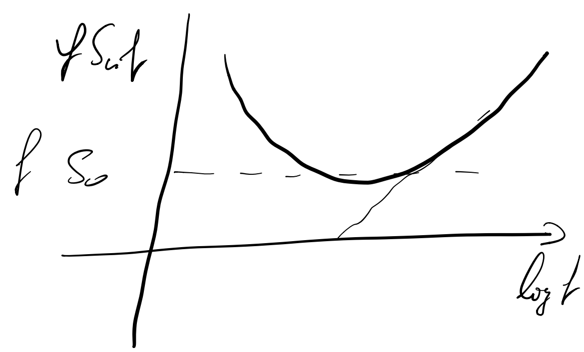
$$\Rightarrow \quad \left(\frac{S}{N}\right)^2 = 4\int_0^\infty df\,\frac{|\hat{h}|^2}{S_n(f)}$$

$$\left(\frac{S}{N}\right)^2 = \frac{5}{6}\,\frac{1}{\pi^{4/3}}\,\frac{\mathcal{M}^{5/3}}{\boxed{D^2}}\,|Q|^2\int_0^{f_{max}} df\,\frac{f^{-7/3}}{S_n(f)}$$

angle average $\langle |Q| \rangle \simeq \frac{2}{5}$

$$\boxed{D \sim \frac{\mathcal{M}_c^{5/6}}{\left(\frac{S}{N}\right)}\sqrt{\int_0^{f_{max}} df\,\frac{f^{-7/3}}{S_n(f)}}}$$

detectable distance scales $\frac{1}{\left(\frac{S}{N}\right)}$!!



Let's do some simple assumptions:

$S \approx S_0 = \text{const}$

$$\Rightarrow \quad \frac{S}{N} \sim \frac{1}{D} \mathcal{M}_c^{5/6} S_0^{-\frac{1}{2}} f_0^{-4/3}$$

Number of cycles: $N_c = \mathcal{M}_c^{-5/3} f_0^{-5/3}$

$h_0 \sim \frac{1}{D} f_0^{2/3} \mathcal{M}_c^{5/3}$ (real space!)

$$\Rightarrow \quad \frac{S}{N} \sim \frac{h_0}{\sqrt{f_0 S_0}} N_c^{1/2} \Leftarrow$$

$\frac{S}{N}$ improves with number of orbits

$\frac{S}{N}$ of a single burst