1

B1913+16

- Discovered in 1974 ; Nobel prize in 1993 to Joe Taylor + Russel Hulse for discovery (not for confirming GR)
- 1st DNS : $M_1 = 1.438\ M_\odot$ $\rightarrow$ $M_c = 1.231\ M_\odot$

  $M_2 = 1.390\ M_\odot$

$P_{orb} = 7.75$ hr (27,900 s)

$a = 1.9501 \times 10^6$ km

What is expected rate of GW decay ?

$$\frac{\dot{P}}{P} = \frac{-\dot{\Omega}}{\Omega} = -\frac{3}{2}\frac{\dot{E}_{orb}}{E_{orb}}$$

$$E_{orb} = \frac{-GM_1M_2}{2a}$$

Plug in Kepler 3 : $\Omega^2 = \left(\frac{2\pi}{P}\right)^2 = \frac{G(M_1+M_2)}{a^3}$

$\hookrightarrow$ $E_{orb} = \frac{-(GM_c)^{5/3}\,\Omega^{2/3}}{2}$, where $M_c = \frac{(M_1M_2)^{3/5}}{(M_1+M_2)^{1/5}}$

Using quadrupole formula result (see Lena's 1st lecture) :

$$-\langle \dot{E}_{orb} \rangle = \frac{1}{5}\langle \dddot{Q}_{ij}\,\dddot{Q}_{ij} \rangle$$

$$= \frac{32}{5}\left(GM_c\,\Omega\right)^{10/3}\frac{1}{c^5}$$

recall: $Q_{xx} = \eta\, r^2\cos^2\phi - \frac{1}{3}$

$Q_{yy} = \eta\, r^2\sin^2\phi - 1/3$

$Q_{xy} = Q_{yx} = \eta\, r^2 \sin\phi\cos\phi$

$\eta = \frac{m_1m_2}{m_T}$, $\phi = \phi_0 + \Omega t$

$$\boxed{\dot{P} = -P\,\frac{3}{2}\left(\frac{\dot{E}_{orb}}{E_{orb}}\right) = -\frac{96}{5c^5}\left(GM_c\right)^{5/3}\Omega^{8/3}\,P}$$

Valid in limit that evolution is slow cf. orbital vel., $\|\dot{a}\| \ll \Omega a$

$\hookrightarrow$ $\dot{P} = -2.024 \times 10^{-13}$ s/s

Observed $\dot{P} = -2.433 \times 10^{-12}$ s/s

$\}$ We are off by ~12x.

What are we missing? $e \rightarrow 0.617$

Peters and Matthews (1963)

For general (non-circ. orbit):

$$r = \frac{a(1-e^2)}{1+e\cos\phi} \quad , \quad \dot{\phi} = \frac{[G(M_1+M_2)a(1-e^2)]^{1/2}}{r^2}$$

Plug this into quad. formula:

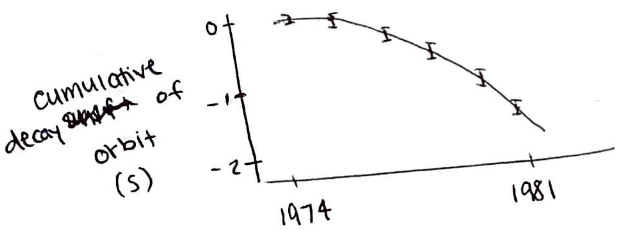
$$\langle \dot{E}_{orb} \rangle = f_0 \langle \dot{E}_{circ.} \rangle$$

where $$f_0 = \frac{1}{(1-e^2)^{7/2}}\left[1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right]$$

$$f_0(e=0.617) = 11.8$$

↳ $\dot{P}_{predicted} \approx \dot{P}_{obs}$

Actual data:

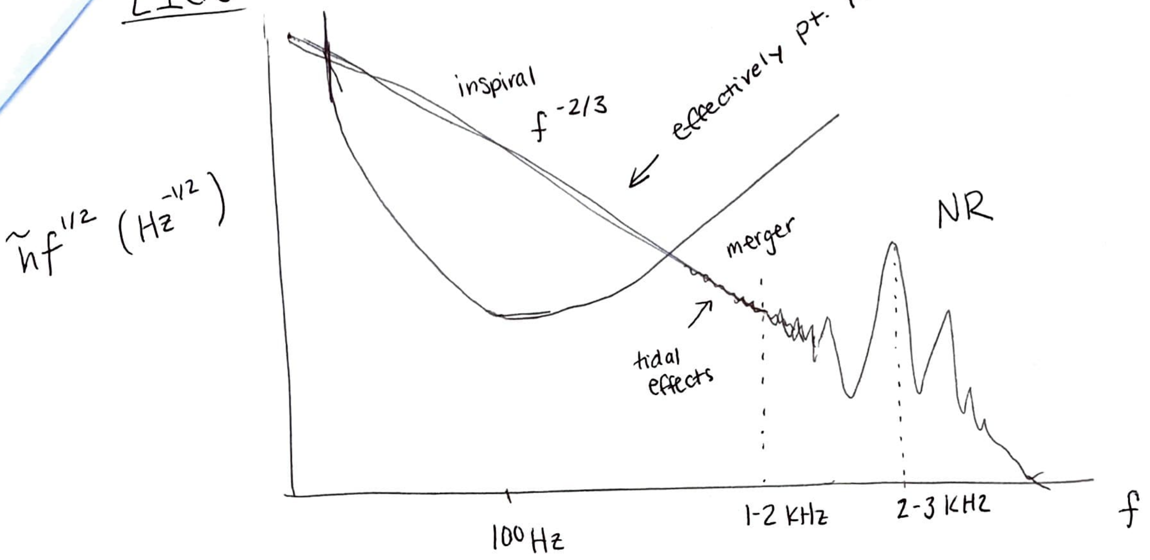


Weisberg + Taylor 1981:
1st measurement of $\dot{P}$, showing 15% accurate match to GR

Taylor + Weisberg 1982 → famous plot

Weisberg + Huang 2016 → update

LIGO



Freq-domain strain (assuming stationary phase approx.):

$$\tilde{h}_{+,\times}(f) = A_{+,\times} \, M_c^{5/6} f^{-7/6} e^{-i\psi}$$

where

$$\psi = 2\pi f t_c - 2\phi_c - \frac{\pi}{4} + \frac{3}{128 \eta x^{5/2}} \left[1 + \psi_{PP-PN}(x,\eta) + \psi_{tidal}(x,\eta,\Lambda_1,\Lambda_2) \right]$$

$\left[M_c = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}} \right.$; $\left. M_c(q=1) = M_1/2^{1/5} \right]$

$$\left.\frac{\text{Amplitude of } 1.4-1.4 M_\odot \text{ BNS}}{\text{Amplitude of } 20-20 M_\odot \text{ BBH}}\right|_{\text{same freq.}} = \left(\frac{1.4}{20}\right)^{5/6} \approx 0.1$$

BNS is <u>~10x weaker</u> than "typical" BBH signal (at same freq.)

How long is BNS signal is LIGO sensitivity range?

$$\frac{1}{t_D} \equiv \frac{\dot{f}}{f} = -\frac{\dot{P}}{P} \rightarrow t_D = \left(\frac{96}{5c^5}\right)^{-1} (GM_c)^{-5/3} (\pi f)^{-8/3}$$

$$t_D \approx 140 \left(\frac{M_c}{1.2 M_\odot}\right)^{-5/3} \left(\frac{f}{30 Hz}\right)^{-8/3} s$$

Once stars reach separation s.t. $f_{GW} = 30$ Hz, takes ~2 min to merge; 20-20 $M_\odot$ BBH merges ~100x faster.

How many orbital cycles is this?

$$N_{cycles} = \int_{t_a}^{t_b} f \, dt = \int_{f_a}^{f_b} \frac{f}{\dot{f}} \, df = \int_{f_a}^{f_b} t_D \, df$$

$$= \frac{c^5}{32\pi \left(G M_c \pi f_a\right)^{5/3}} \left[1 - \left(\frac{f_a}{f_b}\right)^{5/3}\right]$$

- reducing low-freq. range significantly incr. # of cycles
- Orig GW170817 analysis incl. down to 30 Hz ; re-analysis went to 23 Hz

$f_a = 30 \to 23\,Hz$ gives $\approx 60\%$ more orbits

23-2040 Hz for GW170817 ($M_c = 1.186\,M_\odot$) $\to$ $N_{cycles} = 4200$

But, this neglects tides!

If $\boxed{\text{tides}}$ are present, this will act to accel. the inspiral:

Consider now:
$$\begin{cases} E_{orb} = E_N + E_{tide} \\ \dot{E}_{orb} = \dot{E}_{GW} + \dot{E}_{tide} \end{cases}$$

$$N_{cycles} = \int \frac{f}{\dot{f}}\, df \underset{E_{orb} \propto (\pi f)^{2/3}}{=} \frac{3}{2}\int \frac{E_{orb}}{\dot{E}_{orb}}\, df = \frac{3}{2}\int \frac{E_N + E_{tide}}{\dot{E}_{GW} + \dot{E}_{tide}}\, df$$

$$= \int \underbrace{\frac{3}{2}\frac{E_N}{\dot{E}_{GW}}}_{\equiv\, t_D} \frac{(1 + E_{tide}/E_N)}{(1 + \dot{E}_{tide}/\dot{E}_{GW})}\, df \longrightarrow (1 + \dot{E}_{tide}/\dot{E}_{GW})^{-1} \approx 1 - \frac{\dot{E}_{tide}}{\dot{E}_{GW}}$$

valid for $\dot{E}_{tide} \ll \dot{E}_{GW}$

Keeping terms to leading order:

$$N = \int t_D \left(1 + \frac{E_{tide}}{E_N} - \frac{\dot{E}_{tide}}{\dot{E}_{GW}}\right) df = N_{GW} + \Delta N$$

This is what we are after. $\rightarrow$
$$\Delta N = \int t_D \left(\frac{E_{tide}}{E_N} - \frac{\dot{E}_{tide}}{\dot{E}_{GW}}\right) df$$

To leading post-Newt. order:

Vines, Flanagan, + Hinderer 2011

$$\begin{cases} E(x) = -\frac{1}{2} M_T \eta\, x \Big[1 + [pN] \overbrace{- 9\frac{M_2}{M_1}\frac{\lambda_1}{M_T^5} x^5 + 1 \leftrightarrow 2}^{\text{tidal terms}} \Big] \\ \dot{E}(x) = -\frac{32}{5}\eta^5 x^5 \Big[1 + [pN] + 6\frac{M_1 + 3M_2}{M_1}\frac{\lambda_1}{M_T^5} x^5 + 1 \leftrightarrow 2 \Big] \end{cases}$$

Setting $M_1 = M_2$, $x = \left(\frac{\pi G M_T f}{c^3}\right)^{2/3}$, $M_T = 2M_1$:

$$E_{tide}/E_N = -\frac{9}{2^5}\frac{\lambda_1}{M_1^5}\left(\frac{2\pi G M_1 f}{c^3}\right)^{10/3} \times 2$$

$$\dot{E}_{tide}/\dot{E}_{GW} = \frac{6(1+3q)}{2^5}\frac{\lambda_1}{M_1^5}\left(\frac{2\pi G M_1 f}{c^3}\right)^{10/3} \times 2$$

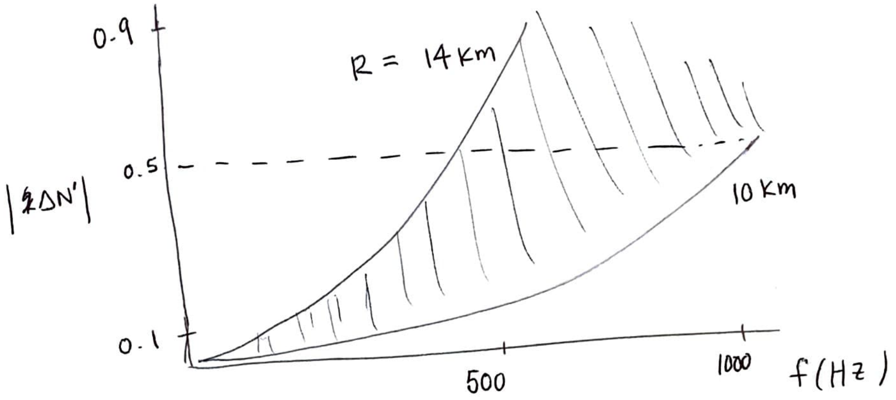
$\leftarrow$ Note signs. Both terms act to accelerate the merger.

Finally, plug in $\lambda_1 = \frac{2}{3} k_2 \left(\frac{c^2 R}{G}\right)^5$

factor of 2 b/c

$$\hookrightarrow \Delta N' \simeq 2\Delta N \approx -0.013 \left(\frac{M}{1.4 M_\odot}\right)^{-10/3} \left(\frac{k_2}{0.26}\right)\left(\frac{R}{10\ \text{km}}\right)^5 \left(\frac{f}{100\ \text{Hz}}\right)^{5/3}$$

$\uparrow$ Newtonian tidal Love #, for $n=1$ polytrope

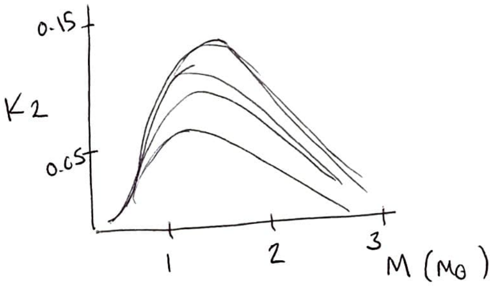


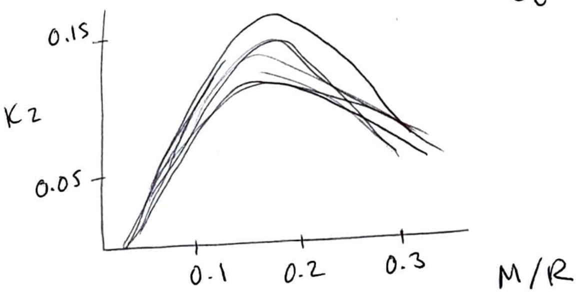
Andersson + Ho 2018

• Larger stars produce a larger tidal effect, but still $\lesssim 1$ cycle

Tidal Love number:

[Hinderer (2010)]





for different theoretical equations of state

For realistic EOS and relativistic calculation, k_2 depends on M and also on details of EOS (i.e., pressure vs. density)

[Cf. Newtonian value of $k_2 = 0.26$ in calculation of ΔN above]

What does LIGO actually measure?

The phase of $\tilde{h}(f) \sim A f^{-7/6} e^{-i\psi}$ is given by the formula:

$$\frac{d^2\psi}{df^2} = \frac{2\, dE/df}{\dot{E}}$$

Taking E_{tidal} and $\dot{E}_{tidal}$ at leading order (see eqs. on p.5) can calculate ψ_{tidal}:

Flanagan & Hinderer 2008

$$\psi_{tidal} = -\frac{9}{16}\frac{x^5}{\eta M_T^5}\left[\left(11\frac{m_2}{m_1} + \frac{M_T}{m_1}\right)\lambda_1 + 1 \leftrightarrow 2\right]$$

$$\psi_{total} = (\ldots) + \frac{3}{128\eta x^{5/2}}\left\{1 + \psi_{PN} \underbrace{- 24\frac{x^5}{M_T^5}\left[\left(11\frac{m_2}{m_1} + \frac{M_T}{m_1}\right)\lambda_1 + 1\leftrightarrow 2\right]}\right.$$

Plug in ~~$\lambda_1 = \frac{2}{3}k_2 R^5$~~

$$\Lambda_1 \equiv \frac{\lambda_1}{m_1^5}:$$

$$-24\frac{x^5}{M_T^5}\left[\left(11\frac{m_2}{m_1} + \frac{M_T}{m_1}\right)\Lambda_1 m_1^5 + 1\leftrightarrow 2\right]$$

$$= -24x^5\left[\frac{11m_2 + (m_1+m_2)}{\cancel{m_1}\, M_T^5}\Lambda_1 m_1^4 + 1\leftrightarrow 2\right]$$

$$= -\frac{39}{2}x^5 \underbrace{\boxed{\frac{16}{13}\left[\frac{(12m_2+m_1)m_1^4\Lambda_1}{(m_1+m_2)^5} + 1\leftrightarrow 2\right]}}$$

$$\equiv \tilde{\Lambda}$$

so that

$$\psi_{total} = (\ldots) + \frac{3}{128\eta x^{5/2}}\left\{1 + \psi_{PN} - \frac{39}{2}\tilde{\Lambda}x^5\right\}$$

Tidal effects formally enter at 5PN, but this term is larger than the currently-known P.P. 5PN terms by $\sim (R/M)^5 \sim 10^5$